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LETTER TO THE EDITOR

A general theory of phase-space quasiprobability distributions

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Abstract. We present a general theory of quasiprobability distributions on phase spaces of quantum systems whose dynamical symmetry groups are (finite-dimensional) Lie groups. The family of distributions on a phase space is postulated to satisfy the Stratonovich–Weyl correspondence with a generalized traciality condition. The corresponding family of the Stratonovich–Weyl kernels is constructed explicitly. In the presented theory we use the concept of generalized coherent states, that brings physical insight into the mathematical formalism.

Since the introduction of the Wigner function in 1932 [1], it has found numerous physical applications. Perhaps the most important is the phase-space formulation of quantum mechanics that has its origins in the early work of Moyal [2]. In this formulation, a function on phase space is associated with an operator on Hilbert space, opening the way to formally representing quantum mechanics as a statistical theory on classical phase space. Various aspects of the phase-space formalism for basic quantum systems have been developed by a number of authors (e.g. see [3–16]). More extensive lists of the literature on the subject can be found in reviews [17–21].

Besides the Wigner function W, other phase-space functions have been considered in the literature. In particular, the Husimi Q function and the Glauber–Sudarshan P function have found extensive applications in quantum optics. Cahill and Glauber [4] have shown that there exists a whole family of phase-space functions parametrized by a number s; the values $+1$, 0, and -1 of s correspond to the Q, W, and P functions, respectively. These phase-space functions are known as quasiprobability distributions (QPDs), as they play in quantum mechanics a role similar to that of genuine probability distributions in classical statistical mechanics.

The phase-space formalism has been applied successfully to the description of a spinless quantum particle and a mode of the quantized radiation field (modelled by a quantum harmonic oscillator). The corresponding phase space is $\mathbb{R}^2$ (or, equivalently, the complex plane $\mathbb{C}$). A generalization of this description to a set of N independent particles or harmonic oscillators in a p-dimensional world is straightforward [21]. A more complicated problem is the phase-space description of spin. A number of authors have used different approaches to the construction of the Wigner function for spin [8, 9, 11, 15, 22–27]. The explicit expressions for the Q, W, and P functions for arbitrary spin were first obtained by

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Agarwal [8], who used the spin coherent-state representation [28–30] and the Fano multipole operators [31]. Várilly and Gracia-Bondía [11] have shown that the spin coherent-state approach is equivalent to a general mathematical formalism based on the Stratonovich–Weyl (SW) correspondence [3] and on the concept of twisted product [7]. This formalism has also been generalized [12] to compact semisimple groups.

In this letter we develop a general theory of QPDs on phase spaces of quantum systems whose dynamical symmetry groups are (finite-dimensional) Lie groups. This theory can be viewed as a generalization of the Cahill–Glauber QPDs (related to the Heisenberg–Weyl group) to arbitrary Lie groups. We make clear that the structure of the family of the phase-space QPDs for a Lie group is determined by the group covariance and the traciality conditions.

Let G be a Lie group (connected and simply connected, with finite dimension n), that is the dynamical symmetry group of a given quantum system. Let T be a unitary irreducible representation of G acting on the Hilbert space $\mathcal{H}$. By choosing a fixed normalized reference state $|\psi_0\rangle \in \mathcal{H}$, one can define the system of coherent states $\{|\psi_g\rangle\}$:

$$|\psi_g\rangle = T(g)|\psi_0\rangle \qquad g \in G. \tag{1}$$

The isotropy subgroup $H \subset G$ consists of all the group elements h that leave the reference state invariant up to a phase factor,

$$T(h)|\psi_0\rangle = \mathrm{e}^{\mathrm{i}\phi(h)}|\psi_0\rangle \qquad |\mathrm{e}^{\mathrm{i}\phi(h)}| = 1, h \in H. \tag{2}$$

For every element $g \in G$, there is a decomposition of g into a product of two group elements, one in H and the other in the coset space $X = G/H$,

$$g = \Omega h \qquad g \in G, h \in H, \Omega \in X. \tag{3}$$

It is clear that group elements g and g' with different h and h' but with the same Ω produce coherent states which differ only by a phase factor: $|\psi_g\rangle = \mathrm{e}^{\mathrm{i}\delta}|\psi_{g'}\rangle$, where $\delta = \phi(h) - \phi(h')$. Therefore, a coherent state $|\Omega\rangle \equiv |\psi_\Omega\rangle$ is determined by a point $\Omega = \Omega(g)$ in the coset space G/H. A very important property is the identity resolution in terms of the coherent states:

$$\int_X \mathrm{d}\mu(\Omega)|\Omega\rangle\langle\Omega| = I \tag{4}$$

where $\mathrm{d}\mu(\Omega)$ is the invariant integration measure on $X = G/H$, the integration is over the whole manifold X, and I is the identity operator on $\mathcal{H}$.

An important class of coherent-state systems corresponds to the coset spaces $X = G/H$ which are homogeneous Kählerian manifolds. Then X can be considered as the phase space of a classical dynamical system. The standard (or maximum-symmetry) systems of the coherent states correspond to the cases when an 'extreme' state of the representation Hilbert space (e.g., the vacuum state of an oscillator or the lowest/highest spin state) is chosen as the reference state. This choice of the reference state leads to systems consisting of states with properties 'closest to those of classical states' [30, 32]. In what follows we will consider the coherent states of maximal symmetry and assume that the phase space of the quantum system is a homogeneous Kählerian manifold $X = G/H$, each point of which corresponds to a coherent state $|\Omega\rangle$. In particular, the Glauber coherent states of the Heisenberg–Weyl group H_3 are defined on the complex plane $\mathbb{C} = \mathrm{H}_3/\mathrm{U}(1)$, and the spin coherent states are defined on the unit sphere $\mathbb{S}^2 = \mathrm{SU}(2)/\mathrm{U}(1)$. In the more rigorous mathematical language of Kirillov's theory [33], the phase space X is the co-adjoint orbit associated with the unitary irreducible representation T of the group G on the Hilbert space $\mathcal{H}$.

The idea of the phase-space formalism is as follows. Let A be an operator on $\mathcal{H}$. Then A can be mapped by a family of functions (quasiprobability distributions) $F_A^{(s)}(\Omega)$ onto the phase space X. (The index s that labels functions in the family will be determined shortly.) The function $F_A^{(s)}(\Omega)$ is called the SW image of A, if it satisfies the properties known as the SW correspondence [3]:

(0) Linearity: $A \to F_A^{(s)}(\Omega)$ is a one-to-one linear map.

(i) Reality:

$$F_{A^\dagger}^{(s)}(\Omega) = [F_A^{(s)}(\Omega)]^*. \tag{5a}$$

(ii) Standardization:

$$\int_X \mathrm{d}\mu(\Omega) F_A^{(s)}(\Omega) = \mathrm{Tr} A. \tag{5b}$$

(iii) Covariance:

$$F_{g\cdot A}^{(s)}(\Omega) = F_A^{(s)}(g^{-1}\Omega) \tag{5c}$$

where $g \cdot A \equiv T(g)AT(g)^{-1}$.

(iv) Traciality:

$$\int_X \mathrm{d}\mu(\Omega) F_A^{(s)}(\Omega) F_B^{(-s)}(\Omega) = \mathrm{Tr}(AB). \tag{5d}$$

These conditions have a clear physical meaning. The linearity and the traciality conditions are related to the statistical interpretation of the theory. If B is the density matrix, then the traciality condition (5d) assures that the statistical average of the phase-space distribution F_A coincides with the quantum expectation value of the operator A. O'Connell and Wigner [34] have shown that the traciality condition for density matrices of a spinless quantum particle (there it appears as an overlap relation) is necessary for the uniqueness of the definition of the Wigner function. It has also been shown [12] that the traciality condition is necessary for the uniqueness of the definition of the symbol calculus (twisted or 'star' products) of the phase-space functions and for the validity of the related non-commutative Fourier analysis. Equation (5d) is actually a generalization of the usual traciality condition [3, 11, 12], as it holds for any s and not only for the Wigner case $s = 0$. The reality condition (5a) means that if A is self-adjoint, then $F_A^{(s)}(\Omega)$ is real. The condition (5b) is a natural normalization, which means that the image of the identity operator I is the constant function 1. The covariance condition (5c) means that the phase-space formulation must explicitly express the symmetry of the system.

The linearity is taken into account, if we implement the map $A \to F_A^{(s)}(\Omega)$ by the generalized Weyl rule

$$F_A^{(s)}(\Omega) = \mathrm{Tr}[A\Delta^{(s)}(\Omega)] \tag{6}$$

where $\{\Delta^{(s)}(\Omega)\}$ is a family (labelled by s) of operator-valued functions on the phase space X. These operators are referred to as the SW kernels. The generalized traciality condition (5d) is taken into account if we define the inverse of the generalized Weyl rule (6) as

$$A = \int_X \mathrm{d}\mu(\Omega) F_A^{(s)}(\Omega) \Delta^{(-s)}(\Omega). \tag{7}$$

Now, the conditions (5a)–(5c) of the SW correspondence for $F_A^{(s)}(\Omega)$ can be translated into the following conditions on the SW kernel $\Delta^{(s)}(\Omega)$:

$$\text{(i)}\ \Delta^{(s)}(\Omega) = [\Delta^{(s)}(\Omega)]^\dagger \qquad \forall\, \Omega \in X. \tag{8a}$$

$$\text{(ii)} \int_X \mathrm{d}\mu(\Omega)\Delta^{(s)}(\Omega) = I. \tag{8b}$$

$$\text{(iii)}\ \Delta^{(s)}(g\Omega) = T(g)\Delta^{(s)}(\Omega)T(g)^{-1}. \tag{8c}$$

Substituting the inverted maps (7) for A and B into the generalized traciality condition (5d), we obtain the relation between the QPDs with different values of the index s:

$$F_A^{(s)}(\Omega) = \int_X \mathrm{d}\mu(\Omega')K_{s,s'}(\Omega,\Omega')F_A^{(s')}(\Omega') \tag{9}$$

$$K_{s,s'}(\Omega,\Omega') \equiv \mathrm{Tr}[\Delta^{(s)}(\Omega)\Delta^{(-s')}(\Omega')]. \tag{10}$$

If we take in equation (9) $s = s'$ and take into account the arbitrariness of A, we obtain the following relation

$$\Delta^{(s)}(\Omega) = \int_X \mathrm{d}\mu(\Omega')K(\Omega,\Omega')\Delta^{(s)}(\Omega') \tag{11}$$

where the function

$$K(\Omega,\Omega') = \mathrm{Tr}[\Delta^{(s)}(\Omega)\Delta^{(-s)}(\Omega')] \tag{12}$$

behaves like the delta function on the manifold X.

Now, our problem is to find the explicit form of the SW kernel $\Delta^{(s)}(\Omega)$ that satisfies the conditions (8a)–(8c) and (11). We start by considering the Hilbert space $L^2(X,\mu)$ of square-integrable functions $u(\Omega)$ on X with the invariant measure $\mathrm{d}\mu$. The representation T of the Lie group G on $L^2(X,\mu)$ is defined as

$$T(g)u(\Omega) = u(g^{-1}\Omega). \tag{13}$$

The eigenfunctions $Y_\nu(\Omega)$ of the Laplace–Beltrami operator [35] form a complete orthonormal basis in $L^2(X,\mu)$:

$$\sum_\nu Y_\nu^*(\Omega)Y_\nu(\Omega') = \delta(\Omega-\Omega') \tag{14a}$$

$$\int_X \mathrm{d}\mu(\Omega)Y_\nu^*(\Omega)Y_{\nu'}(\Omega) = \delta_{\nu\nu'}. \tag{14b}$$

The functions $Y_\nu(\Omega)$ are called the harmonic functions, and $\delta(\Omega-\Omega')$ is the delta function in X with respect to the measure $\mathrm{d}\mu$. (The index ν is multiple; it has one discrete part, while the other part is discrete for compact manifolds and continuous for non-compact manifolds. In the latter case the summation over ν includes an integration with the Plancherel measure and the symbol $\delta_{\nu\nu'}$ includes some Dirac delta function. For more details see [35]. For conciseness, we omit these details in our formulae.) The eigenfunctions $Y_\nu(\Omega)$ are linear combinations of matrix elements $T_{\nu\nu'}(g)$. Therefore, the transformation rule for the harmonic functions is [35]

$$T(g)Y_\nu(\Omega) = Y_\nu(g^{-1}\Omega) = \sum_{\nu'} T_{\nu'\nu}(g)Y_{\nu'}(\Omega). \tag{15}$$

The function $|\langle\Omega|\Omega'\rangle|^2$ is symmetric in Ω and Ω'. Therefore, its expansion in the orthonormal basis must be of the form

$$|\langle\Omega|\Omega'\rangle|^2 = \sum_\nu \tau_\nu Y_\nu^*(\Omega)Y_\nu(\Omega') = \sum_\nu \tau_\nu Y_\nu^*(\Omega')Y_\nu(\Omega) \tag{16}$$

where τ_ν are real positive coefficients. Since $|\langle\Omega|\Omega'\rangle|^2$ is real and $Y_\nu^*(\Omega) = \mathrm{e}^{\mathrm{i}\phi(\nu)}Y_{\tilde\nu}(\Omega)$, the coefficients τ_ν must be invariant under this index transformation: $\tau_\nu = \tau_{\tilde\nu}$. Since $\langle\Omega|\Omega'\rangle = \langle g\Omega|g\Omega'\rangle$, the coefficients τ_ν must be invariant under the index transformation of equation (15): $\tau_\nu = \tau_{\nu'}$ for the discrete part of ν and $\mathrm{d}\rho(\nu)\tau(\nu) = \mathrm{d}\rho(\nu')\tau(\nu')$ for the continuous part of ν, where $\mathrm{d}\rho(\nu)$ is the Plancherel measure.

Let us now define the set of operators $\{D_\nu\}$ on $\mathcal{H}$:

$$D_\nu \equiv \omega_\nu \int_X \mathrm{d}\mu(\Omega) Y_\nu(\Omega) |\Omega\rangle\langle\Omega| \tag{17}$$

where ω_ν are real coefficients to be determined from the normalization condition. Using expression (16), we obtain the orthogonality condition

$$\mathrm{Tr}(D_\nu D_{\nu'}^\dagger) = (\tau_\nu \omega_\nu^2)\delta_{\nu\nu'}. \tag{18}$$

The proper normalization is then obtained by taking

$$\omega_\nu^2 = 1/\tau_\nu. \tag{19}$$

Note that ω_ν is defined only up to a sign, $\omega_\nu = \pm\tau_\nu^{-1/2}$, which determines the sign of D_ν. Using (16), we also obtain the relation

$$\omega_\nu \langle\Omega|D_\nu|\Omega\rangle = Y_\nu(\Omega). \tag{20}$$

The coefficients ω_ν satisfy the same invariance conditions as τ_ν (up to a choice of the sign). Therefore, D_ν are the tensor operators whose transformation rule is the same as for the harmonic functions $Y_\nu(\Omega)$:

$$T(g) D_\nu T(g)^{-1} = \sum_{\nu'} T_{\nu'\nu}(g) D_{\nu'}. \tag{21}$$

An operator A on $\mathcal{H}$ can be expanded in the orthonormal basis $\{D_\nu\}$:

$$A = \sum_\nu \mathrm{Tr}(A D_\nu^\dagger) D_\nu. \tag{22}$$

Now we are able to find the SW kernel $\Delta^{(s)}(\Omega)$ with all the desired properties. Specifically, let us define

$$\Delta^{(s)}(\Omega) \equiv \sum_\nu f(s;\tau_\nu) Y_\nu^*(\Omega) D_\nu. \tag{23}$$

We will see that the SW kernel (23) is a generalization of the Cahill–Glauber kernel for a harmonic oscillator [4, 5] and the Agarwal kernel for spin [8]. We show that the construction of the generalized kernel (23) satisfies the SW correspondence. In equation (23) $f(s;\tau_\nu)$ is a function of τ_ν and of the index s. We assume that f possesses the invariance properties of τ_ν. The reality condition (8*a*) is then satisfied if $f(s;\tau_\nu)$ is a real-valued function. Therefore, we can consider only real values of the index s. Then it is sufficient to use the convention in which $s \in [-1, 1]$. (Note that this restriction of the range of s is not necessary. For example, Leonhardt and Paul [36] have shown that the Cahill–Glauber QPDs with real s are of experimental relevance for realistic homodyne measurements. Wünsche [37] has generalized the Cahill–Glauber QPDs by introducing the complete Gaussian class of quasiprobabilities characterized by a three-dimensional complex vector parameter. Experimental applications of these distributions have been recently discussed [38].)

Next we consider the standardization condition (8*b*). Using the identity resolution (4) and equation (16), we can write

$$1 = \langle\Omega|\Omega\rangle = \int_X \mathrm{d}\mu(\Omega') |\langle\Omega|\Omega'\rangle|^2 = \sum_\nu \tau_\nu Y_\nu^*(\Omega) \int_X \mathrm{d}\mu(\Omega') Y_\nu(\Omega'). \tag{24}$$

Multiplying the left- and right-hand sides of this equation by $Y_{\nu'}(\Omega)$ and integrating over $\mathrm{d}\mu(\Omega)$, we obtain

$$\int_X \mathrm{d}\mu(\Omega) Y_\nu(\Omega) = \tau_\nu \int_X \mathrm{d}\mu(\Omega) Y_\nu(\Omega). \tag{25}$$

Since τ_ν is not identically 1, this relation can be satisfied only if there exists some ν_0 such that $\tau_{\nu_0} = 1$ and

$$\int_X \mathrm{d}\mu(\Omega) Y_\nu(\Omega) \propto \delta_{\nu\nu_0}. \tag{26}$$

(As was already mentioned, for non-compact manifolds the symbol $\delta_{\nu\nu'}$ actually includes some Dirac delta functions.) Then the standardization condition (8*b*) is satisfied if

$$f(s;1) = \omega_{\nu_0} = \pm 1 \qquad \forall s. \tag{27}$$

The covariance condition (8*c*) is guaranteed by virtue of the transformation rules (15) and (21) and by the invariance of τ_ν under these index transformations.

In order to satisfy the relation (11), the function $K(\Omega, \Omega')$ of equation (12) must be the delta function in X with respect to the measure $\mathrm{d}\mu$,

$$K(\Omega, \Omega') = \sum_\nu Y_\nu^*(\Omega) Y_\nu(\Omega') = \delta(\Omega - \Omega'). \tag{28}$$

This result is valid if

$$f(s;\tau_\nu) f(-s;\tau_\nu) = 1. \tag{29}$$

This property is satisfied only by the exponential function, i.e.

$$f(s;\tau_\nu) = \pm[f(\tau_\nu)]^s. \tag{30}$$

Note that the standardization condition (27) then reads $f(1) = 1$. The double-valuedness of type (30) was pointed out by Várilly and Gracia-Bondía [11] who considered the Wigner function for spin. The exact form of the function $f(\tau_\nu)$ can be determined if we define for $s = 1$

$$\Delta^{(1)}(\Omega) \equiv |\Omega\rangle\langle\Omega|. \tag{31}$$

Then we obtain $\pm f(\tau_\nu) = 1/\omega_\nu = \pm\tau_\nu^{1/2}$, i.e.

$$f(\tau_\nu) = \sqrt{\tau_\nu}. \tag{32}$$

Obviously, the standardization condition $f(1) = 1$ is satisfied. This result concludes the construction of the generalized SW kernel. It is evident that the properties of the kernels are completely determined by the harmonic functions on the corresponding manifold and by the coherent states that form this manifold. We also note that the function $K_{s,s'}(\Omega, \Omega')$ of equation (10) is given by

$$K_{s,s'}(\Omega, \Omega') = \sum_\nu \tau_\nu^{(s-s')/2} Y_\nu^*(\Omega) Y_\nu(\Omega') \tag{33}$$

and it clearly satisfies condition (9).

In order to illustrate the general formalism, we show how it works on two basic examples. First, we consider a spinless quantum particle whose dynamical symmetry group is the Heisenberg–Weyl group H_3. The corresponding nilpotent Lie algebra is spanned by the basis $\{a, a^\dagger, I\}$, where a and $a^\dagger$ are the boson annihilation and creation operators, satisfying the canonical commutation relation. The phase space is the complex plane $\mathbb{C} = H_3/U(1)$, and the (Glauber) coherent states are $|\Omega\rangle \equiv |\alpha\rangle = D(\alpha)|0\rangle$, $\alpha \in \mathbb{C}$, where $D(\alpha) = \exp(\alpha a^\dagger - \alpha^* a)$ is the displacement operator. The invariant measure is

$d\mu(\Omega) \equiv \pi^{-1} d^2\alpha$, and the corresponding delta function is $\delta(\Omega - \Omega') \equiv \pi\delta^{(2)}(\alpha - \alpha')$. The harmonic functions on $\mathbb{C}$ are the exponentials: $Y_\nu(\Omega) \equiv Y_\xi(\alpha) \equiv Y(\xi, \alpha) = \exp(\xi\alpha^* - \xi^*\alpha)$. Here $\nu \equiv \xi \in \mathbb{C}$ with the Plancherel measure $d\rho(\nu) \equiv \pi^{-1} d^2\xi$ and with $\delta_{\nu,\nu'} \equiv \pi\delta^{(2)}(\xi - \xi')$. We find that the invariant coefficient is $\tau_\nu \equiv \tau(\xi) = \exp(-|\xi|^2)$, and the tensor operator

$$D_\nu \equiv D(\xi) = e^{|\xi|^2/2} \int_{\mathbb{C}} \frac{d^2\alpha}{\pi} e^{\xi\alpha^* - \xi^*\alpha} |\alpha\rangle\langle\alpha| \tag{34}$$

is just the displacement operator $D(\xi) = e^{\xi a^\dagger - \xi^* a}$. Then one obtains the SW kernel:

$$\Delta^{(s)}(\alpha) = \int_{\mathbb{C}} \frac{d^2\xi}{\pi} e^{-s|\xi|^2/2} e^{\xi^*\alpha - \xi\alpha^*} e^{\xi a^\dagger - \xi^* a} \tag{35}$$

which is exactly the Cahill–Glauber kernel $T(\alpha, -s)$ [4].

The second example is spin whose dynamical symmetry group is SU(2). The corresponding simple Lie algebra is spanned by the basis $\{J_+, J_-, J_3\}$. The unitary irreducible representations are labelled by the index j $(j = 0, 1/2, 1, \ldots)$, and the Hilbert space $\mathcal{H}_j$ is spanned by the orthonormal basis $|j, m\rangle (m = j, j-1, \ldots, -j)$. The phase space is the unit sphere $\mathbb{S}^2 = SU(2)/U(1)$, and the coherent states are $|\Omega\rangle \equiv |j; \theta, \phi\rangle = \exp(\beta J_+ - \beta^* J_-)|j, -j\rangle$, where $\beta = \frac{1}{2}\theta\, e^{-i\phi}$. The invariant measure is $d\mu(\Omega) \equiv (4\pi)^{-1}(2j+1)\sin\theta\, d\theta\, d\phi$, and the corresponding delta function is $\delta(\Omega - \Omega') \equiv 4\pi/(2j+1)\delta(\cos\theta - \cos\theta')\delta(\phi - \phi')$. The harmonic functions on $\mathbb{S}^2$ are the familiar spherical harmonics: $Y_\nu(\Omega) = \sqrt{4\pi/(2j+1)} Y_{lm}(\theta, \phi)$. Here ν is the double discrete index $\{l, m\}$ with $l = 0, 1, 2, \ldots$ and $m = l, l-1, \ldots, -l$. We find that the invariant coefficient is independent of m: $\tau_\nu \equiv \tau_l = \langle j, j; l, 0|j, j\rangle^2$, where $\langle j_1, m_1; j_2, m_2|j, m\rangle \equiv C^{j_1 j_2 j}_{m_1 m_2 m}$ is the Clebsch–Gordan coefficient. Note that $\tau_l = 0$ for $l > 2j$. The tensor operator is the well known Fano multipole operator [31], which can be written in the form

$$D_{lm} = \sqrt{\frac{2l+1}{2j+1}} \sum_{r,s=-j}^{j} \langle j, r; l, m|j, s\rangle |j, s\rangle\langle j, r|. \tag{36}$$

Then the SW kernel is

$$\Delta^{(s)}(\theta, \phi) = \sqrt{\frac{4\pi}{2j+1}} \sum_{l=0}^{2j} \langle j, j; l, 0|j, j\rangle^s \sum_{m=-l}^{l} D_{lm} Y^*_{lm}(\theta, \phi), \tag{37}$$

as was found by Agarwal [8] and by Várilly and Gracia-Bondía [11].

As the explicit form of the SW kernels is known, we can write the generalized QPDs on the phase space as

$$F_A^{(s)}(\Omega) = \sum_\nu \tau_\nu^{s/2} \mathcal{A}_\nu Y_\nu(\Omega) \tag{38}$$

$$\mathcal{A}_\nu \equiv \mathrm{Tr}(A D_\nu^\dagger) = \omega_\nu \int_X d\mu(\Omega) Y_\nu^*(\Omega) \langle\Omega|A|\Omega\rangle. \tag{39}$$

In particular, for $s = 1$, we obtain the Q function (Berezin's covariant symbol [6]):

$$Q_A(\Omega) \equiv F_A^{(1)}(\Omega) = \langle\Omega|A|\Omega\rangle. \tag{40}$$

For $s = -1$, we obtain the P function (Berezin's contravariant symbol [6]):

$$P_A(\Omega) \equiv F_A^{(-1)}(\Omega) = \sum_\nu \omega_\nu \mathcal{A}_\nu Y_\nu(\Omega) \tag{41}$$

$$A = \int_X d\mu(\Omega) P_A(\Omega) |\Omega\rangle\langle\Omega|. \tag{42}$$

The functions P and Q are counterparts in the traciality condition (5d). Perhaps the most important QPD corresponds to $s = 0$, because this function is 'self-conjugate' in the sense that it is the counterpart of itself in the traciality condition (5d). It is natural to call the QPD with $s = 0$ the generalized Wigner function:

$$W_A(\Omega) \equiv F_A^{(0)}(\Omega) = \sum_{\nu} \mathcal{A}_{\nu} Y_{\nu}(\Omega). \tag{43}$$

In conclusion, we have developed the general group-theoretical formalism of the phase-space QPDs. Starting from a number of physically sensible basic postulates (the SW correspondence), we have explicitly constructed the SW kernel that implements the bijective transformation between phase-space functions and Hilbert-space operators for quantum systems with general Lie-group symmetries. More details and examples of the QPDs on phase spaces of physical systems will be presented elsewhere.

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